An alternative estimator of a probability of failure

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Problematic

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
- Let $\mathbf{X}$ be a **random variable**: $\mathbf{X} : \Omega \to \mathbb{R}^d$ and $\mathbf{X} \sim \mathrm{P}_{\mathbf{X}}$.
- Let $g : \mathbb{R}^d \to \mathbb{R}$ be a measurable function of $\mathbf{X}$.

Goal

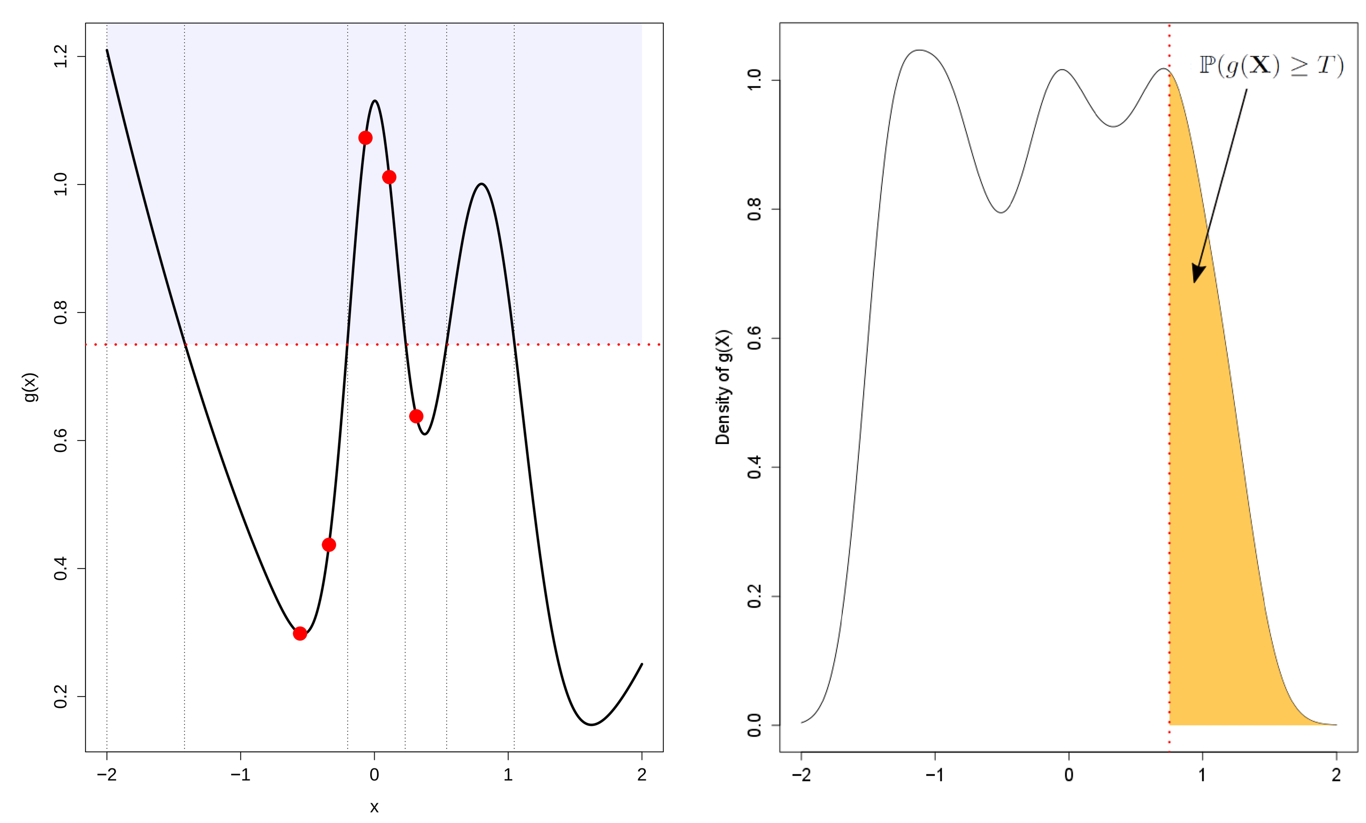
Estimate the **probability** p of $g(\mathbf{X})$ exceeding a threshold T:

$$p = \mathbb{P}(g(\mathbf{X}) \geq T) = \int_{\mathbb{R}^d} \mathbb{1}_{g(\mathbf{x}) \geq T} \mathrm{P}_{\mathbf{X}}(d\mathbf{x}),$$

typically called the **failure probability**.

Framework

- The function g is an **expensive-to-evaluate black-box** function.
- A small number of observations: $\mathcal{D}_n = \{(\mathbf{x}_1, g(\mathbf{x}_1)), \ldots, (\mathbf{x}_n, g(\mathbf{x}_n))\}$.
- ☹ Crude Monte Carlo estimator can not be used.



Left: The true function g. The threshold T=0.75 ⋯. Only 5 observations • are available. **Right**: The target failure probability.

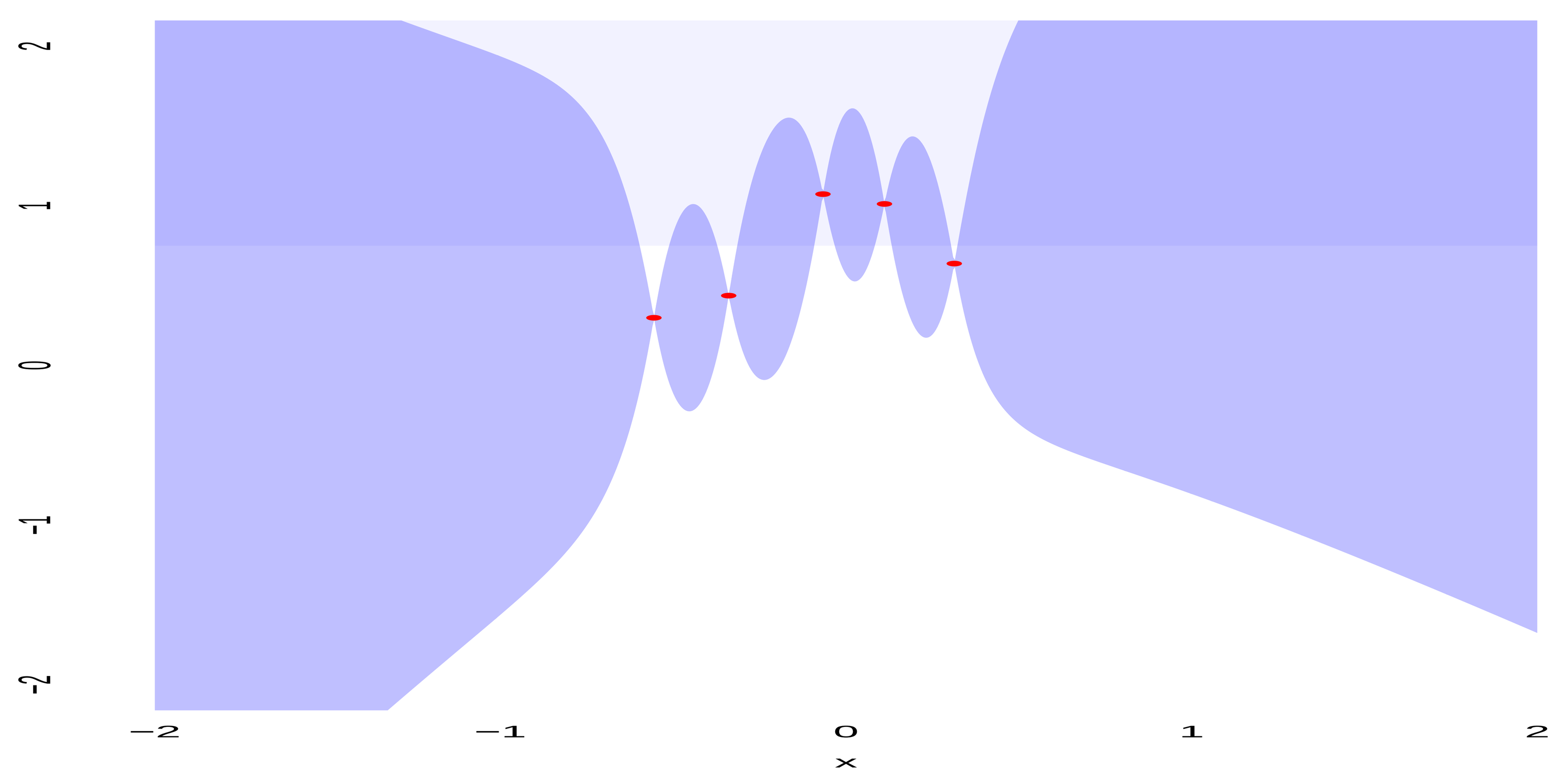
Gaussian Process regression

- The function g is a **realization of a Gaussian process** $\xi = \{\xi(\mathbf{x})\}_{\mathbf{x} \in \mathbb{R}^d}$.
- The prior process ξ **conditioned on the observed data** $\mathcal{D}_n$ is the **posterior process** ξ_n and is still **Gaussian**:

$$\forall\ \mathbf{x} \in \mathbb{R}^d, \quad \xi_n(\mathbf{x}) = [\xi(\mathbf{x}) \mid \xi(\mathbf{x}_i) = g(\mathbf{x}_i)_{1 \leq i \leq n}] \sim \mathcal{N}(m_n(\mathbf{x}), \sigma_n^2(\mathbf{x})),$$

where $m_n(\mathbf{x})$ and $\sigma_n^2(\mathbf{x})$ are explicitly known.

- Each realization of ξ_n is an **interpolation model** for g.



The true function g. Only 5 observations • are available. Some realizations of the process ξ_n. Its mean function ⋯.

A first estimation

- As a result, p is a **realization of the random variable** $P_n : \Omega \to [0, 1]$ defined by

$$P_n = \mathbb{P}(\xi_n(\mathbf{X}) \geq T \mid \xi_n) = \int_{\mathbb{X}} \mathbb{1}_{\xi_n(\mathbf{x}) \geq T} \mathrm{P}_{\mathbf{X}}(d\mathbf{x}).$$

Estimator of p

- A **natural estimator** $\widehat{p}_n$ **of** p is the **mean value** of P_n:

$$\widehat{p}_n = \mathbb{E}[P_n] = \int_{\mathbb{X}} \mathbb{E}\left[\mathbb{1}_{\xi_n(\mathbf{x}) \geq T}\right] \mathrm{P}_{\mathbf{X}}(d\mathbf{x}) = \mathbb{E}\left[s_n(\mathbf{X})\right],$$

where $s_n(\mathbf{x}) = \mathbb{E}\left[\mathbb{1}_{\xi_n(\mathbf{x}) \geq T}\right] = \mathbb{P}(\xi_n(\mathbf{x}) \geq T) = \Phi\left((m_n(\mathbf{x}) - T)/\sigma_n(\mathbf{x})\right)$.

- ☺ The estimator $\widehat{p}_n$ is easy to compute.
- ☹ Distribution of P_n is **untraceable**. Computing realizations of P_n is **time consuming** and could lead **numerical issues**.
- **For all** $\alpha \in [0, 1]$**, how to determine the** α**-quantile** $F_{P_n}^{-1}(\alpha)$ **of** P_n**?**

An alternative estimation

- Let $U : \Omega \to [0, 1]$ be a **random variable**.
- We introduce the **random variable** $R_n = R_n(U)$ defined by

$$R_n = \mathbb{P}(s_n(\mathbf{X}) > U \mid U) = \int_{\mathbb{X}} \mathbb{1}_{s_n(\mathbf{x}) > U} \ \mathrm{dP}_{\mathbf{X}}(\mathbf{x}).$$

Properties of R_n

1. The distribution of R_n measurably depends on the distribution of U.

2. $U \sim \mathcal{U}[0, 1] \Leftrightarrow \mathbb{E}[P_n] = \mathbb{E}[R_n]$.

The **mean value** of R_n is an **estimator** of p.

3. $U \sim \mathcal{U}[0, 1] \Leftrightarrow P_n \leq_{cx} R_n$.

- $P_n \leq_{cx} R_n$ means that for all **convex function** φ,

$$\boxed{\mathbb{E}[\varphi(P_n)] \leq \mathbb{E}[\varphi(R_n)].}$$

$\hookrightarrow \mathrm{Var}[P_n] \leq \mathrm{Var}[R_n]$.

The variance of R_n is an **upper bound** of the **variance** of P_n.

$$\hookrightarrow F_{P_n}^{-1}(\alpha) \leq \frac{1}{1-\alpha} \int_\alpha^1 F_{P_n}^{-1}(t)\,dt \leq \frac{1}{1-\alpha} \int_\alpha^1 F_{R_n}^{-1}(t)\,dt \leq \frac{\mathbb{E}\left[P_n\right]}{1-\alpha}.$$

- $1/(1-\alpha) \int_\alpha^1 F_{R_n}^{-1}(t)dt$: the **Conditional Value at Risk** ($\mathrm{CVaR}_\alpha(R_n)$).
- $\mathrm{CVaR}_\alpha(R_n)$ provides a **smaller upper bound** of $F_{P_n}^{-1}(\alpha)$ than the one obtained with Markov inequality.
- ☺ From the **realizations of** R_n, we easily estimate $\mathrm{CVaR}_\alpha(R_n)$and provide **credible intervals of** P_n.

References

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